Radiative Tail in Elastic Electron Scattering

L. C. MAXIMON* AND D. B. ISABELLE†
National Bureau of Standards, Washington, D. C.
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The Bethe-Heitler cross section with arbitrary form factor is integrated over all photon angles, without approximation. The limit of this integrated cross section for large energies ϵ of the initial and final electron is considered in detail. It is shown that the contribution to this cross section from photons emitted in the direction of either the incoming or the outgoing electron gives the terms of order $\ln\epsilon$ found by Schiff, but that the contribution from all other photon directions is of relative order 1, and we give explicit expressions for these terms, as well as a numerical evaluation for some typical cases, since for the energies of experimental interest, viz., 30–1000 MeV, the logarithm is not very large: $\ln\epsilon\approx 4$–8. Furthermore, keeping the terms of order 1 is of particular importance for electron scattering angles ϑ near 180°, since, as is shown, the terms in $\ln\epsilon$ all have the factor $\cos^2\frac{1}{2}\vartheta$, which is not the case for the terms of order 1. It is shown that all the formulas given are also valid for ϑ equal or very close to 180°.

I. INTRODUCTION

In the last few years, excitation of nuclei by inelastic electron scattering has become a very important tool for the study of nuclear spectroscopy,[1] and, as new linear accelerators which will permit one to obtain very accurate data are becoming available, it becomes important to be able to analyze the data with the greatest possible precision. It is well known[2] that one of the limitations in this analysis is the fact that scattered electrons can also lose energy by means of secondary processes. These are due to the following facts: First, nuclear scattering is always accompanied by emission of photons; second, since the target has a finite thickness ($\sim 1/100$ of a radiation length), ionization or emission of a photon in the field of another nucleus may occur before or after the nuclear scattering. We would like to focus our attention on the first process only, as the other two are much easier to deal with, as can be seen in the literature.[1,2]

In 1952 Schiff[3] performed an integration of the Bethe-Heitler cross section for bremsstrahlung[4] in the Coulomb field of a point nucleus, assuming that the photon is emitted in the direction of either the incoming or the outgoing electron, and retaining only the terms of order $\ln\epsilon$, where ϵ is the energy of the electron in units of mc^2.

In this paper we have performed the integration of the Bethe-Heitler cross section with arbitrary form factor (whether for the nucleus or for the atom) over all photon angles without making any approximations, after which we consider in detail the limit of this integrated cross section for large energies of the initial and final electron, $\epsilon_1\gg 1$, $\epsilon_2\gg 1$. We show that the contribution to this cross section from photons emitted in the direction of either the incoming or the outgoing electron gives the terms of order $\ln\epsilon$ found by Schiff, but that the contribution from all other photon directions is of relative order 1, and we give explicit expressions for these terms, as well as numerical examples for some typical cases, since for the energies of experimental interest, viz., 30–1000 MeV, the logarithm is not very large: $\ln\epsilon\approx 4$–8. Furthermore, keeping the terms of order 1 is of particular importance for electron scattering angles ϑ near 180°, since, as will be seen, the terms in $\ln\epsilon$ all have the factor $\cos^2\frac{1}{2}\vartheta$, which is not the case for the terms of order 1. We wish to distinguish, in our discussion of the derived formulas, two aspects of its application to the experimental data: the radiative correction and the radiative tail. The radiative correction has to do with the emission and reabsorption of virtual photons and the emission of real soft photons, and involves an integration over photon energy. The radiative tail appears because of the emission of real hard photons, and is an extension of the scattering peak (elastic or inelastic), differential in the energy of the scattered electron. The problem of the radiative correction has been discussed extensively. (See, e.g., Ref. 5. This paper contains references to previous articles on the same subject.) We therefore consider here only the radiative tail. Nuclear recoil corrections are neglected in the calculation, but their order of magnitude is discussed at the end of the paper. We will treat the problem in first Born approximation throughout. The errors introduced thereby may be expected to be of the same order of magnitude as those introduced by considering the same approximation in large angle Coulomb scattering, for which it has been noticed[6] recently that the error is of order $(Z/137)2\sin\frac{1}{2}\vartheta$,

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† On leave of absence from Centre National de la Recherche Scientifique: Laboratoire de l'Accélérateur Linéaire, Orsay (France).

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[6] J. W. Motz has made a plot of the theoretical ratio of the first Born approximation to the exact cross section for elastic scattering of electrons in a pure Coulomb field (Mott scattering) for several values of the electron energy ϵ, nuclear charge Z, and scattering angle ϑ (private communication). We find that for $\epsilon\gg 1$, all the data for $30°\leq\vartheta\leq 120°$ and $13\leq Z\leq 79$ can be fit by the simple expression $\sigma_{\rm Born}=(1-\delta)\sigma_{\rm exact}$, in which, to within 10%, $\delta=(Z/137)2\sin\frac{1}{2}\vartheta=(Z/137)(\Delta p/p)$, p being the electron momentum, Δp the momentum transfer.

where ϑ is the scattering angle of the electron and Z the nuclear charge. This is clearly the most serious approximation in the entire analysis, and is the next point that must be considered in any refinement of this work.

II. INTEGRATION OF THE CROSS SECTION OVER PHOTON DIRECTIONS

The Born approximation cross section for the scattering of an electron of initial energy and momentum ϵ_1, $\mathbf{p}_1$, final energy and momentum ϵ_2, $\mathbf{p}_2$, and emission of a photon of energy and momentum k, $\mathbf{k}$ (the Bethe-Heitler cross section) is

$$d\sigma=\frac{1}{(2\pi)^2}\frac{e^2}{\hbar c}\left(\frac{Ze^2}{mc^2}\right)^2\frac{p_2}{p_1}\frac{dk}{k}\frac{\mathfrak{F}^2(q)}{q^4}\sin\vartheta_k d\vartheta_k d\phi_k \sin\vartheta d\vartheta d\phi$$
$$\times\left\{\frac{p_1^2\sin^2\theta_1(4\epsilon_2^2-q^2)}{(\epsilon_1-p_1\cos\theta_1)^2}+\frac{p_2^2\sin^2\theta_2(4\epsilon_1^2-q^2)}{(\epsilon_2-p_2\cos\theta_2)^2}\right.$$
$$-\frac{2p_1p_2\sin\theta_1\sin\theta_2\cos(\varphi_1-\varphi_2)(4\epsilon_1\epsilon_2-q^2+2k^2)}{(\epsilon_1-p_1\cos\theta_1)(\epsilon_2-p_2\cos\theta_2)}$$
$$\left.+\frac{2k^2(p_1^2\sin^2\theta_1+p_2^2\sin^2\theta_2)}{(\epsilon_1-p_1\cos\theta_1)(\epsilon_2-p_2\cos\theta_2)}\right\}. \quad (1)$$

We assume here that we are dealing with scattering from a spherically symmetric static charge distribution, so that $\mathfrak{F}(q)$, the nuclear form factor, is a function of the magnitude q of the momentum transfer to the nucleus, and not of its direction. For a deformed charge distribution, the form factor depends on the relative orientation of the charge deformation and the vector $\mathbf{q}$. (See, e.g., Refs. 7 and 8.) Spin and polarization states of the final particles have been summed over, and the average over spin states of the initial electron has been taken. Here (θ_1,φ_1) and (θ_2,φ_2) are the polar and azimuthal angles of the initial and final electron, respectively, in a coordinate system with z axis along the direction of the photon $\mathbf{k}$. Further, (ϑ_k,ϕ_k) and (ϑ,ϕ) are the polar and azimuthal angles of the photon and final electron, respectively, in a coordinate system with z axis along the direction of the initial electron $\mathbf{p}_1$. These angles are related by

$$\theta_1=\vartheta_k\,,$$
$$\cos\theta_2=\cos\vartheta\cos\vartheta_k+\sin\vartheta\sin\vartheta_k\cos(\phi-\phi_k)\,, \quad (2)$$
$$\cos\vartheta=\cos\theta_1\cos\theta_2+\sin\theta_1\sin\theta_2\cos(\varphi_1-\varphi_2)\,,$$

and

$$\mathbf{q}=\mathbf{p}_1-\mathbf{p}_2-\mathbf{k}\,,\quad k=\epsilon_1-\epsilon_2\,. \quad (3)$$

The units of energy and momentum are mc^2 and mc throughout.

Since the emitted photon is not observed, we wish to integrate over the angles of the photon direction, ϑ_k and ϕ_k. In view of the factor $\mathfrak{F}^2(q)$ in the cross section, the convenient variables for this integration are clearly ϑ_k and q^2. Using (2) and (3) we express the cross section in terms of ϑ_k, q^2, ϑ and ϕ. (In addition, we must multiply the cross section by two, since the entire range of q is covered by letting ϕ_k go from 0 to π.) The integration over ϑ_k is "straightforward but tedious."[9] The integration over q^2 is not carried out explicitly at this point, so that we arrive now at the cross section for the scattering of the electron through an angle ϑ:

$$d\sigma=\frac{1}{2\pi}\frac{e^2}{\hbar c}\left(\frac{Ze^2}{mc^2}\right)^2\frac{p_2}{p_1}\frac{dk}{k}\sin\vartheta d\vartheta d\phi\int_{q_m^2}^{q_M^2}\frac{d(q^2)}{q^4}\mathfrak{F}^2(q)$$
$$\times\left\{-\frac{2k}{(2\lambda+k^2)^{1/2}}-k\left(\frac{1}{D_1^{1/2}}-\frac{1}{D_2^{1/2}}\right)\right.$$
$$\times\left(\frac{q^4+4\lambda^2-4q^2(\epsilon_1^2+\epsilon_2^2-1)-16\epsilon_1\epsilon_2}{2\lambda-q^2}\right)$$
$$+2k\frac{(4\epsilon_2^2-q^2)}{D_1^{3/2}}[2\lambda(\lambda-k\epsilon_2)-(\lambda+k\epsilon_1)q^2]$$
$$\left.-2k\frac{(4\epsilon_1^2-q^2)}{D_2^{3/2}}[2\lambda(\lambda+k\epsilon_1)-(\lambda-k\epsilon_2)q^2]\right\}. \quad (4)$$

$$\lambda=\epsilon_1\epsilon_2-p_1p_2\cos\vartheta-1=\tfrac{1}{2}(|\mathbf{p}_1-\mathbf{p}_2|^2-k^2)\,,$$
$$D_1=\{p_1(q^2-q_2^2)+2p_2\lambda_0\cos\vartheta\}^2+4k^2p_2^2\sin^2\vartheta\,,$$
$$D_2=\{p_2(q^2-q_1^2)+2p_1\lambda_0\cos\vartheta\}^2+4k^2p_1^2\sin\vartheta\,,$$
$$q_m=|\mathbf{p}_1-\mathbf{p}_2|-k=(2\lambda+k^2)^{1/2}-k\,, \quad (5)$$
$$q_M=|\mathbf{p}_1-\mathbf{p}_2|+k=(2\lambda+k^2)^{1/2}+k\,,$$
$$q_2=2p_2\sin\tfrac{1}{2}\vartheta\,,$$
$$q_1=2p_1\sin\tfrac{1}{2}\vartheta\,,$$
$$\lambda_0=\epsilon_1\epsilon_2-p_1p_2-1\,.$$

Equation (4) will form the basis for our further considerations. We note at this point, however, that the integrations performed in going from Eq. (1) to Eq. (4), in the course of which we have made no approximations, are also applicable to the scattering of an electron, with emission of a photon, in which the momentum transfer to the nucleus is "small," i.e., $q\lesssim O(1)$. In this case we must replace $\mathfrak{F}(q)$ by $1-F(q)$, where $F(q)$ is the atomic form factor. We note further that for the case of a pure Coulomb potential ($\mathfrak{F}(q)=1$, $F(q)=0$), the integration over q^2 may be performed explicitly, and was in fact first carried out by Racah.[10] We give the result for this

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case, presenting it in a somewhat more condensed form than that appearing elsewhere[10]:

$$d\sigma=\frac{1}{2\pi}\frac{e^2}{\hbar c}\left(\frac{Ze^2}{mc^2}\right)^2\frac{p_2}{p_1}\frac{dk}{k}\sin\vartheta d\vartheta d\phi\left\{2\frac{k^2\lambda+(2\epsilon_1\epsilon_2-\lambda)(\lambda+1)}{\lambda^2[\lambda(\lambda+2)]^{1/2}}\ln\{\lambda+1+[\lambda(\lambda+2)]^{1/2}\}\right.$$
$$+\frac{k}{\lambda^2p_2^3}\left[-2k\epsilon_2p_2^2+3\epsilon_2(\epsilon_1+\epsilon_2)-(p_1^2+p_2^2)+(2\epsilon_1\epsilon_2-\lambda)(\epsilon_1\epsilon_2+p_2^2)+\frac{2\epsilon_2p_1^2\sin^2\vartheta(2\epsilon_1p_2^2-3k\epsilon_2^2)}{\lambda^2}\right]\ln(\epsilon_2+p_2)$$
$$-\frac{k}{\lambda^2p_1^3}\left[2k\epsilon_1p_1^2+3\epsilon_1(\epsilon_1+\epsilon_2)-(p_1^2+p_2^2)+(2\epsilon_1\epsilon_2-\lambda)(\epsilon_1\epsilon_2+p_1^2)+\frac{2\epsilon_1p_2^2\sin^2\vartheta(2\epsilon_2p_1^2+3k\epsilon_1^2)}{\lambda^2}\right]\ln(\epsilon_1+p_1)$$
$$+\frac{2k^2\sin^2\vartheta}{\lambda^4p_1^2p_2^2}[2p_1^2p_2^2(\epsilon_1^2+\epsilon_2^2-\epsilon_1\epsilon_2)+3k^2(\epsilon_1+\epsilon_2)^2]+\frac{k^2}{\lambda^2p_1^2p_2^2}[(p_1^2+p_2^2)(\epsilon_1\epsilon_2+1)$$
$$\left.-(2\epsilon_1\epsilon_2-\lambda)(p_1^2+p_2^2+\epsilon_1\epsilon_2+1)-3(\epsilon_1+\epsilon_2)^2]-\frac{2}{\lambda^2}[2\epsilon_1\epsilon_2-\lambda]\right\}. \quad (6)$$

III. HIGH-ENERGY LIMIT OF THE CROSS SECTION

We now return to Eq. (4) and perform the high-energy approximations pertinent to the experiments we are considering, viz., $\epsilon_1\gg1$, $\epsilon_2\gg1$. In addition, we assume throughout all but the last section of the paper that $\sin\vartheta$ is of order 1, by which we mean $1/(\epsilon\sin\vartheta)^2\ll1$, i.e., that ϑ is not very close to either 0 or π. At the end of the paper the case in which ϑ is close to or equal to π, $(\pi-\vartheta\lesssim O(1/\epsilon))$, is discussed, and it is shown that all of the formulas derived for $\sin\vartheta$ of order one remain equally valid for $\pi-\vartheta\lesssim O(1/\epsilon)$. With these assumptions we have, from the expressions (5),

$$\begin{aligned}D_1&\approx\epsilon_1^2(q^2-q_2^2)^2+4k^2\epsilon_2^2\sin^2\vartheta,\\D_2&\approx\epsilon_2^2(q^2-q_1^2)^2+4k^2\epsilon_1^2\sin^2\vartheta,\\\lambda&\approx2\epsilon_1\epsilon_2\sin^2\tfrac{1}{2}\vartheta.\end{aligned}\quad (7)$$

Thus, since q^2 is, throughout the range of integration, of order ϵ^2, D_1 and D_2 will be of order ϵ^6, except near the points $q^2=q_2^2$ (for D_1) and $q^2=q_1^2$ (for D_2), at which points they are much smaller, of order ϵ^4. Thus at these points the integrand in expression (4) will be sharply peaked; a simple calculation shows that for $\sin\vartheta$ of order unity, $q_m<q_2<q_1<q_M$, i.e., the peaks lie within the range of integration in expression (4).

We now consider in detail the expression in curly brackets in the cross section (4), the various terms of which are shown in Fig. 1. We note first that the terms with factor $D_1^{-1/2}$ or $D_2^{-1/2}$ have peaks of the same height and width as do the terms with the factor $D_1^{-3/2}$ or $D_2^{-3/2}$. Thus, all the peaked terms should be expected to give contributions of equal order of magnitude to the cross section. Further, we note that, for the terms with the factor $D_1^{-3/2}$ or $D_2^{-3/2}$, the ratio of the height of the peaks to that of the background is of order ϵ^3, whereas the ratio of the width of the peaks to that of the total integration region is of order $1/\epsilon$. Thus, for these terms the background gives a negligible contribution, of order $1/\epsilon^2$ relative to that of the peaks, and may be neglected. This is not the case, however, for the terms with factor $D_1^{-1/2}$ or $D_2^{-1/2}$. For these terms the ratio of peak height to that of the background is only of order ϵ, while the ratio of the width of the peaks to the total integration region is of order $1/\epsilon$, as before. Thus the contribution of the background may be expected to be of the same order of magnitude as that of the peaks for these terms. Finally, the one term without any peaked factor, denoted by D^0 in Fig. 1, is of the same order of magnitude as are the terms with factor $D_1^{-1/2}$ or $D_2^{-1/2}$ in the background region, and hence must also be kept. These order of magnitude considerations are indeed borne out in the explicit evaluations which follow, except in that the contribution from the peaks of the terms with factor $D_1^{-1/2}$ or $D_2^{-1/2}$ are actually of order $\ln\epsilon$ relative to the other contributions we have mentioned, i.e., these order of magnitude considerations do not distinguish between $\ln\epsilon$ and 1, but only between different powers of ϵ. Noting, however, that even for $\epsilon=10^3$ we have $\ln\epsilon=6.9$,

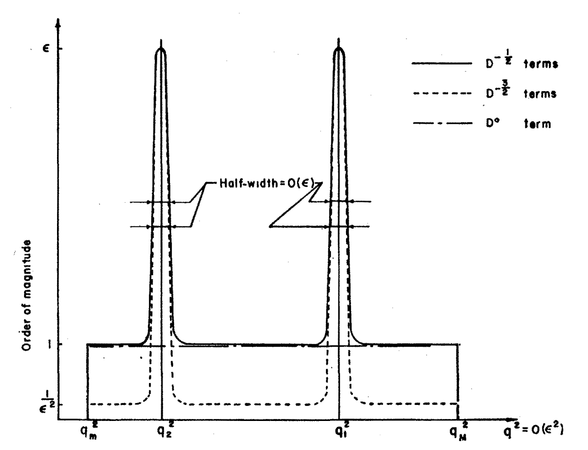

FIG. 1. Relative order of magnitude representation of the various terms in the integrand of cross section (4). For detailed explanation see text.

we keep, throughout, terms of relative order one as well as those of order $\ln\epsilon$, neglecting only terms of relative order $1/\epsilon^2$.

We now separate the contribution of the peaks from that of the background, writing explicit expressions for each of these contributions. Consider the terms with factor $D_1^{-1/2}$ or $D_2^{-1/2}$ first. We have, then, integrals of the form

$$I=\int_{x_m}^{x_M}\frac{f(x)dx}{[(x-x_0)^2+\eta^2]^{1/2}},\tag{8}$$

where

$$x_m<x_0<x_M$$

and (9)

$$0<\eta\ll x_M-x_0,\quad 0<\eta\ll x_0-x_m.$$

We write

$$I=f(x_0)\int_{x_m}^{x_M}\frac{dx}{[(x-x_0)^2+\eta^2]^{1/2}}+\int_{x_m}^{x_M}\frac{f(x)-f(x_0)}{[(x-x_0)^2+\eta^2]^{1/2}}dx.\tag{10}$$

The first integral on the right-hand side of (10) is the contribution from the peak at $x=x_0$, the second integral is the background contribution. In evaluating the first integral we neglect terms of relative order $\eta^2/(x_M-x_0)^2$ and $\eta^2/(x_0-x_m)^2$, which are, for $\sin\vartheta$ of order one, of order $1/\epsilon^2$ for the integrals in expression (4). In the second integral there is no singularity at $x=x_0$ and hence we may set $\eta=0$, again introducing errors of relative order $1/\epsilon^2$. Thus we obtain

$$I=f(x_0)\ln\left[\frac{4(x_M-x_0)(x_0-x_m)}{\eta^2}\right]+\int_{x_m}^{x_M}\frac{f(x)-f(x_0)}{|x-x_0|}dx.\tag{11}$$

For the terms with factor $D_1^{-3/2}$ or $D_2^{-3/2}$ we proceed in similar fashion. We now have integrals of the form

$$J=\int_{x_m}^{x_M}\frac{f(x)dx}{[(x-x_0)^2+\eta^2]^{3/2}}\tag{12}$$

which we write in the form

$$J=f(x_0)\int_{x_m}^{x_M}\frac{dx}{[(x-x_0)^2+\eta^2]^{3/2}}+\int_{x_m}^{x_M}\frac{f(x)-f(x_0)}{[(x-x_0)^2+\eta^2]^{3/2}}dx.\tag{13}$$

The explicit evaluation of the first integral, with neglect, as before, of terms of relative order $1/\epsilon^2$, gives $2/\eta^2$, i.e., a contribution of order ϵ^2 relative to that of the second integral, as previously indicated. Thus we obtain

$$J=\frac{2}{\eta^2}f(x_0).\tag{14}$$

Using the expressions for q_m and q_M in Eqs. (5) and Eqs. (7)–(14), we find, for the high-energy limit of expression (4),

$$d\sigma=\frac{1}{2\pi}\frac{e^2}{\hbar c}\left(\frac{Ze^2}{mc^2}\right)^2\frac{dk}{k}\frac{\sin\vartheta d\vartheta d\phi}{\epsilon_1^2}\{P+B\}.\tag{15}$$

Here the contribution P from the peaks, is

$$P=\frac{\cos^2\frac{1}{2}\vartheta}{\sin^4\frac{1}{2}\vartheta}\left\{\mathfrak{F}^2(q_2)\frac{(\epsilon_1^2+\epsilon_2^2)}{2\epsilon_2^2}\ln(2\epsilon_1)+\mathfrak{F}^2(q_1)\frac{(\epsilon_1^2+\epsilon_2^2)}{2\epsilon_1^2}\ln(2\epsilon_2)-\mathfrak{F}^2(q_2)\frac{\epsilon_1}{2\epsilon_2}-\mathfrak{F}^2(q_1)\frac{\epsilon_2}{2\epsilon_1}\right\},\tag{16}$$

and the contribution from the background B is found, after some algebraic juggling, to be

$$\begin{aligned}B=&k\int_{q_1^2}^{q_M^2}[\epsilon_1G_1(q)-\epsilon_2G_2(q)]a(q)d(q^2)-k\int_{q_m^2}^{q_2^2}[\epsilon_1G_1(q)-\epsilon_2G_2(q)]a(q)d(q^2)+\frac{\epsilon_1\epsilon_2}{2\lambda}\int_{q_2^2}^{q_1^2}[G_1(q)-G_2(q)]b(q)d(q^2)\\&+\frac{1}{\lambda^2}\mathfrak{F}^2(q_1)\left\{k\epsilon_2[k^2\cos^2\tfrac{1}{2}\vartheta-\epsilon_2(\epsilon_1+\epsilon_2)\sin^2\tfrac{1}{2}\vartheta]+2\epsilon_1\epsilon_2[k^2+2\epsilon_1\epsilon_2\cos^2\tfrac{1}{2}\vartheta]\ln\sin\tfrac{1}{2}\vartheta+2\epsilon_2^2[\epsilon_1^2+\epsilon_2^2\cos^2\tfrac{1}{2}\vartheta]\ln\frac{\epsilon_1}{\epsilon_2}\right\}\\&+\frac{1}{\lambda^2}\mathfrak{F}^2(q_2)\left\{-k\epsilon_1[k^2\cos^2\tfrac{1}{2}\vartheta-\epsilon_1(\epsilon_1+\epsilon_2)\sin^2\tfrac{1}{2}\vartheta]+2\epsilon_1\epsilon_2[k^2+2\epsilon_1\epsilon_2\cos^2\tfrac{1}{2}\vartheta]\ln\sin\tfrac{1}{2}\vartheta+2\epsilon_1^2[\epsilon_2^2+\epsilon_1^2\cos^2\tfrac{1}{2}\vartheta]\ln\frac{\epsilon_2}{\epsilon_1}\right\}\\&+\frac{2k\epsilon_1\epsilon_2}{(2\lambda+k^2)^{1/2}}\int_{q_m^2}^{q_M^2}\{1-\mathfrak{F}^2(q)\}\frac{d(q^2)}{q^4}-\frac{2k^2\epsilon_1\epsilon_2}{\lambda^2},\end{aligned}\tag{17}$$

where

$$G_{1,2}(q)=\frac{\mathfrak{F}^2(q)-\mathfrak{F}^2(q_{1,2})}{q^2-q_{1,2}^2}, \qquad \lambda=2\epsilon_1\epsilon_2\sin^2\tfrac{1}{2}\vartheta$$
$$b(q)=\frac{4\lambda^2+q^4-4q^2(\epsilon_1^2+\epsilon_2^2)}{q^4}, \quad a(q)=\frac{b(q)}{2\lambda-q^2}. \tag{18}$$

We have written the integrals in B in such a way that they vanish for $\mathfrak{F}(q)=1$, and that the lack of a singularity in the integrand at $q^2=2\lambda\approx q_1q_2$, $(q_2^2<2\lambda<q_1^2)$, is clear.

IV. DISCUSSION OF THE FORMULAS

It is worth noting, from Eqs. (16) and (17), that the terms in P are of order $\ln\epsilon$ and order 1, and that, as our order of magnitude considerations showed, those in B are all of order 1 since we consider ϵ_1/ϵ_2, the argument of the logarithm in B, to be of order 1. However, these terms will add to those in P and hence change the argument of the logarithmic terms in P. Thus, if one retains only the contribution from the peaks, the coefficients of terms in $\ln\epsilon$ are given correctly, but *not* the argument of the logarithm. For example, for the case $\mathfrak{F}(q)=1$ we have $G_1(q)=G_2(q)=0$ and

$$P=\frac{(\epsilon_1^2+\epsilon_2^2)\cos^2\frac{1}{2}\vartheta}{2\epsilon_1^2\epsilon_2^2\sin^4\frac{1}{2}\vartheta}[\epsilon_1^2\ln(2\epsilon_1)+\epsilon_2^2\ln(2\epsilon_2)-\epsilon_1\epsilon_2],$$

$$B=\frac{1}{4\epsilon_1^2\epsilon_2^2\sin^4\frac{1}{2}\vartheta}\left\{k^2(\epsilon_1^2+\epsilon_2^2)(\sin^2\tfrac{1}{2}\vartheta-\cos^2\tfrac{1}{2}\vartheta)+4\epsilon_1\epsilon_2[k^2+2\epsilon_1\epsilon_2\cos^2\tfrac{1}{2}\vartheta]\ln\sin\tfrac{1}{2}\vartheta-2(\epsilon_1^4-\epsilon_2^4)\cos^2\tfrac{1}{2}\vartheta\ln\frac{\epsilon_1}{\epsilon_2}\right\}, \tag{19}$$

and substituting these in Eq. (15) we find the cross section for the case of a point charge:

$$d\sigma=\frac{1}{2\pi}\frac{e^2}{\hbar c}\left(\frac{Ze^2}{mc^2}\right)^2\frac{dk}{k}\frac{\sin\vartheta d\vartheta d\phi}{\epsilon_1^2}\left\{\frac{(\epsilon_1^2+\epsilon_2^2)}{2\epsilon_2^2}\ln(2\epsilon_2)\frac{\cos^2\frac{1}{2}\vartheta}{\sin^4\frac{1}{2}\vartheta}+\frac{(\epsilon_1^2+\epsilon_2^2)}{2\epsilon_1^2}\ln(2\epsilon_1)\frac{\cos^2\frac{1}{2}\vartheta}{\sin^4\frac{1}{2}\vartheta}+\frac{k^2+2\epsilon_1\epsilon_2\cos^2\frac{1}{2}\vartheta}{\epsilon_1\epsilon_2\sin^4\frac{1}{2}\vartheta}\ln\sin\tfrac{1}{2}\vartheta\right.$$
$$\left.+\frac{(\epsilon_1^2+\epsilon_2^2)}{4\epsilon_1^2\epsilon_2^2\sin^4\frac{1}{2}\vartheta}[k^2\sin^2\tfrac{1}{2}\vartheta-(\epsilon_1^2+\epsilon_2^2)\cos^2\tfrac{1}{2}\vartheta]\right\}. \tag{20}$$

Comparing the logarithmic terms in Eq. (20) with those in Eq. (16), we note that the result of adding the background terms has been the interchange of the arguments of these terms, for the case $\mathfrak{F}(q)=1$. The fact that Eq. (20) may be obtained directly by taking the high energy limit of Eq. (6) provides a check on these calculations.

The idea employed in the evaluation of the high-energy limit of the cross section, namely the separation of the contribution of the peaks from the background contribution in the integral over the momentum transfer, has been described clearly by Schiff.[3] He notes that the differential cross section will have peaks when the photon is emitted very nearly in the direction of either the incident or final electron. That these two cases correspond, respectively, to our peaks at $q^2=q_2^2$ and $q^2=q_1^2$ may be seen quite simply: If $\mathbf{k}$ is in the direction of $\mathbf{p}_1$, then $\mathbf{p}_1-\mathbf{k}=(p_1-k)\hat{p}_1\approx p_2\hat{p}_1$ and $\mathbf{q}=\mathbf{p}_1-\mathbf{p}_2-\mathbf{k}\approx p_2(\hat{p}_1-\hat{p}_2)$, so that $q^2\approx 4p_2^2\sin^2\frac{1}{2}\vartheta=q_2^2$. If $\mathbf{k}$ is in the direction of $\mathbf{p}_2$, then $\mathbf{p}_2+\mathbf{k}=(p_2+k)\hat{p}_2\approx p_1\hat{p}_2$ and $\mathbf{q}=\mathbf{p}_1-\mathbf{p}_2-\mathbf{k}\approx p_1(\hat{p}_1-\hat{p}_2)$, so that $q^2\approx 4p_1^2\sin^2\frac{1}{2}\vartheta=q_1^2$.

Schiff, however, keeps only the contributions of relative order $\ln\epsilon$, as he states explicitly, and these come, as we have seen, only from the peaks. He has done the calculation with $\mathfrak{F}(q)=1$, but it is very easy, following his derivation, to introduce the factors $\mathfrak{F}(q_1)$ and $\mathfrak{F}(q_2)$ which appear in expression (15), and this is in fact what is done by experimentalists[2,11] using his formula. Thus we conclude that the results of Schiff are valid to the approximation which he claims, namely, keeping only the contribution of order $\ln\epsilon$. The principal difference between our high-energy cross section and that of Schiff is thus that he neglects terms of relative order $1/\ln\epsilon$, whereas we neglect only terms of order $1/\epsilon^2$. We see, moreover, that to achieve this we must keep contributions from the background as well as from the peaks. We may note that for the case $\mathfrak{F}(q)=1$, one can obtain Schiff's result directly from the high energy approximation to Racah's[10] cross section, expression (17). However, without Schiff's analysis of the peaks, it would not be clear how to generalize to the case $\mathfrak{F}(q)\neq 1$.

The corrections to the Bethe-Heitler differential cross section due to the recoil of the nucleus, of charge Z and

[11] J. I. Friedman, Phys. Rev. **116**, 1257 (1959).

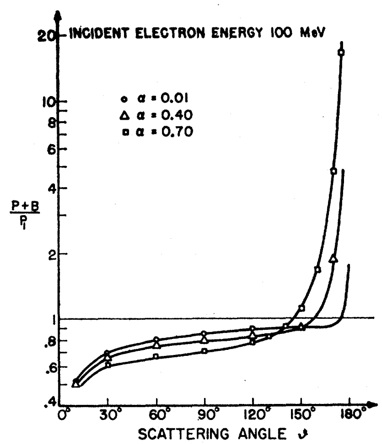

FIG. 2. The ratio $(P+B)/P_1$ as a function of the scattering angle for different values of $\alpha \equiv k/\epsilon_1$, and for 100-MeV incident electron energy, P_1 being the sum of the logarithmic terms in P [the first two terms in Eq. (16)] and representing the terms calculated by Schiff.

mass M, have been calculated by Drell.[12] He shows that there are kinematic correction terms of order qm/M which modify the peaked terms (of order $\ln\epsilon$), and dynamic corrections (due to emission of the bremsstrahlung gamma ray by the nucleus) of order Zqm/M. These latter terms do not, however, correlate strongly the direction of the emitted photon with the direction of the incident or final electron, and hence have a factor of order 1 only, as we have shown. Thus, relative to the cross section we have computed, the kinematic corrections are estimated to be of order $\epsilon m/M$ for large scattering angles, whereas the dynamic corrections should be of relative order $(\epsilon/\ln\epsilon)(Zm/M)$. We note that this last term is roughly independent of Z. Since these corrections are, for the energies of experimental interest (30–1000 MeV), smaller than those introduced by using the Born approximation, we do not include them in this calculation.

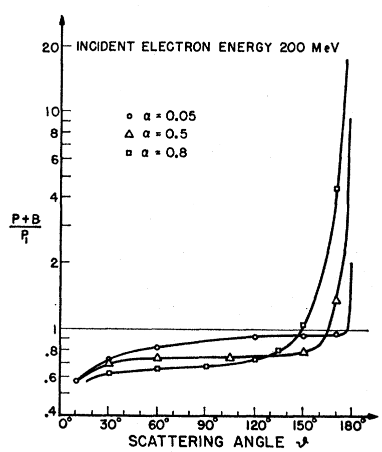

FIG. 3. The ratio $(P+B)/P_1$ as a function of the scattering angle for different values of $\alpha \equiv k/\epsilon_1$, and for 200-MeV incident electron energy, P_1 being the sum of the logarithmic terms in P [the first two terms in Eq. (16)] and representing the terms calculated by Schiff.

To show the importance of the corrections due to the fact that we have done the integrations taking into account the contribution from the background as well as from the peaks, we have plotted in Figs. 2 and 3 the ratio of the complete expression, $P+B$ [Eqs. (16) and (17)], to the sum of the logarithmic terms in P [the first two terms in Eq. (16)], which are the ones given by Schiff's calculation. We have done this for the case of oxygen 16 using for the form factor the expression which gives the best experimental fit[7,13]:

$$\mathfrak{F}(q) = (1-2.58\times10^{-6}q^2)\exp(-5.55\times10^{-6}q^2), \quad (18)$$

where q is expressed in mc units. The ratio has been calculated for two different primary electron energies, 100 and 200 MeV, and for several values of the energy loss k as a function of the scattering angle. In Figs. 4 and 5, also using the form factor given in (18), we plot

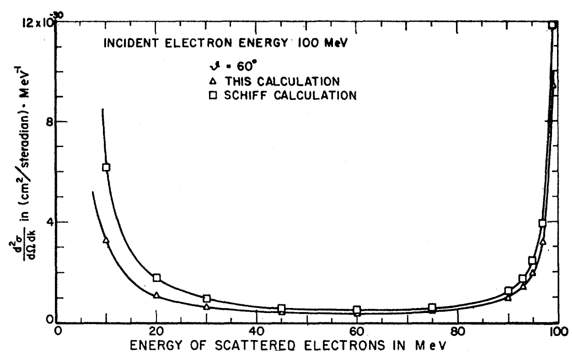

FIG. 4. The cross section calculated from Eq. (15) (this calculation) and the cross section calculated by including only the logarithmic terms in P (Schiff calculation) for incident electron energy 100 MeV and scattering angle 60°.

the cross section as given by Eq. (15) as well as the one obtained by including only the logarithmic terms in P [the first two terms in Eq. (16)], as a function of the final electron energy for fixed initial energy and fixed scattering angle.

V. THE CASE OF 180° SCATTERING ANGLE

Finally, we consider the case in which the scattering angle ϑ is very close or equal to 180°: $\pi-\vartheta\approx\sin\vartheta \lesssim O(1/\epsilon)$. We will see at the end of our considerations that in fact all of the high-energy formulas [in particular expressions (15)–(20)] derived for large $\sin\vartheta[1/(\epsilon\sin\vartheta)^2 \ll 1]$ are equally valid for ϑ very close or equal to 180°. However, many of the statements leading to these formulas must be modified. We return, therefore, to the exact Born approximation cross section given by Eqs. (4) and (5), where we observe, from the expressions for D_1 and D_2, that the peaks actually occur at $q_2'^2$

[12] S. D. Drell, Phys. Rev. **87**, 753 (1952).

[13] F. Lacoste and G. R. Bishop, Nucl. Phys. **26**, 511 (1961).

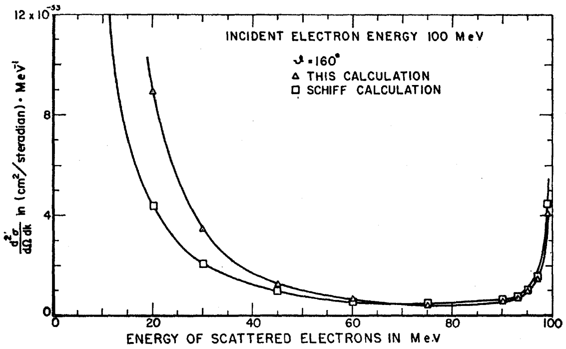

FIG. 5. The cross section calculated from Eq. (15) (this calculation) and the cross section calculated by including only the logarithmic terms in P (Schiff calculation) for incident electron energy 100 MeV and scattering angle 160°.

$\equiv q_2^2-2(p_2/p_1)\lambda_0\cos\vartheta$ and $q_1'^2\equiv q_1^2-2(p_1/p_2)\lambda_0\cos\vartheta$, i.e., they are displaced by quantities of order 1 from q_2^2 and q_1^2 (which are of order ϵ^2). More significant is the fact that as $\sin\vartheta\rightarrow 0$, the peak width, of order $k\sin\vartheta$, goes to zero, and the peak height becomes infinite. However, from Eqs. (5) we find that as ϑ approaches π, the end points of the integration region, q_m^2 and q_M^2, move in toward the peaks at $q_2'^2$ and $q_1'^2$, and that for $\pi-\vartheta$ of order $1/\epsilon$ we find

$$q_2'^2-q_m^2\approx(k\epsilon_2^2/(\epsilon_1+\epsilon_2))[\sin^2\vartheta-(\epsilon_1+\epsilon_2)^2/(\epsilon_1\epsilon_2)]=O(1)$$

and

$$q_M^2-q_1'^2\approx(k\epsilon_1^2/(\epsilon_1+\epsilon_2))[\sin^2\vartheta-(\epsilon_1+\epsilon_2)^2/(\epsilon_1\epsilon_2)]=O(1).$$

Thus, for $\sin\vartheta<(\epsilon_1+\epsilon_2)/(\epsilon_1\epsilon_2)=O(1/\epsilon)$, the peaks at $q_2'^2$ and $q_1'^2$ lie outside the integration region.

We now reconsider the expression in curly brackets in the cross section (4) for $\pi-\vartheta\lesssim O(1/\epsilon)$. The background terms are unaffected by the order of magnitude of $\sin\vartheta$. Thus, as shown in Fig. 1, the background height is of order one for the terms with factors $D_1^{-1/2}$, $D_2^{-1/2}$ or D^0, and must be kept. Terms with factor $D_1^{-3/2}$ or $D_2^{-3/2}$ have background height of order $1/\epsilon^2$ and may be neglected. However, the width of the peak within the integration region is, for $\pi-\vartheta\lesssim O(1/\epsilon)$, of order one rather than of order ϵ, and the peak height (taken at the limits of the integration region if the peaks lie outside these limits) is of order 1 rather than of order ϵ as before, because of the factors of $D_1^{-1/2}$, $D_2^{-1/2}$, $D_1^{-3/2}$ and $D_2^{-3/2}$. Thus, the contribution from the peaks is now of order $1/\epsilon^2$ relative to the background, and may be neglected. The important point is that although the contribution from the peaks is smaller for $\pi-\vartheta\lesssim O(1/\epsilon)$ than for $\sin\vartheta=O(1)$ by a factor of order $1/\epsilon^2$, the contribution from the background is of the same order of magnitude for both cases, and that, our calculation having already included the background contribution, it thus remains valid for both cases. The background terms may now be simplified as was done before in going from Eq. (10) to Eq. (11), where we neglected the η^2 in the denominator of the integrand, with error of relative order $1/\epsilon^2$. With similar error we may now replace $q_2'^2$ and $q_1'^2$ by q_2^2 and q_1^2, respectively, since we noted that $q_2'^2-q_2^2=O(1)$, $q_1'^2-q_1^2=O(1)$. Thus, for $\sin\vartheta>(\epsilon_1+\epsilon_2)/(\epsilon_1\epsilon_2)$, we arrive at precisely those background terms given in expression (17). For $\sin\vartheta<(\epsilon_1+\epsilon_2)/(\epsilon_1\epsilon_2)$, i.e., $q_m^2>q_2'^2$ and $q_M^2<q_1'^2$, the first two integrals in Eq. (17) do not appear, and the third integral should be modified so that the limits of integration are q_m^2 and q_M^2. However, in this case the first two integrals are in fact of order $1/\epsilon^2$ relative to B itself and the change of limits on the third integral would again introduce errors of order $1/\epsilon^2$ relative to B. Thus we can in fact leave the expression (17) for B just as it stands. Likewise, expression (16) for the peak terms may be left without modification, since it is manifestly of order $1/\epsilon^2$ relative to B for $\pi-\vartheta\lesssim O(1/\epsilon)$. Equations (15)–(20) thus need no modification for the case in which $\pi-\vartheta\lesssim O(1/\epsilon)$.

We may mention that a calculation of the cross section for high-energy bremsstrahlung in electron-proton collisions, integrated over the directions of the final proton and photon, has been performed by Berg and Lindner.[14] However, in their calculation the integration is carried out numerically for specific values of the energies and scattering angle and a particular form factor, so that the errors involved in the Schiff approximation are not investigated. Our goal here is a general discussion of the approximations involved in the calculation of Schiff.

After completion of this work, it was brought to our attention that the general problem of the radiation tail has been considered recently by another author,[15] but both the techniques and the goals of his calculation differ from ours.

ACKNOWLEDGMENTS

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[14] R. A. Berg and C. N. Lindner, Phys. Rev. **112**, 2072 (1958).

[15] Yung-Su Tsai, in *Proceedings of the International Conference on Nuclear Structure, Stanford, California*, edited by R. Hofstadter and L. I. Schiff (Stanford University Press, Stanford, California, 1963).